Exercice n°8 : Barrière épaisse en approximation "BKW."

$$(e) \Leftrightarrow \varphi''(x) + \frac{2m}{\hbar^2}\big(E - V(x)\big)\,\varphi(x) = 0$$

avec $V(x)$ d'allure proposée.

① $\varphi(x) = f(x)\, e^{j\theta(x)}$

(H) $f'' \ll$ autres termes

a) $E_c = E - V(x) = \frac{p^2}{2m} \Rightarrow |p| = \sqrt{2m(E - V(x))}$

On injecte $\varphi(x)$ dans l'ESIT :

$$\varphi(x) = f(x)\, e^{j\theta(x)}$$

$$\varphi'(x) = f'(x)\, e^{j\theta(x)} + j\theta'(x)\, f(x)\, e^{j\theta(x)}$$

$$\varphi''(x) = f''(x)\, e^{j\theta(x)} + j f'(x)\theta'(x)\, e^{j\theta(x)} + \big(j\theta''(x) f(x) + j\theta'(x) f'(x)\big) e^{j\theta(x)} + \big(j\theta'(x)\big)^2 f(x)\, e^{j\theta(x)}$$

d'où : (e) $\Leftrightarrow$ $\left[(f'' - \theta'^2 f) + j(2f'\theta' + \theta'' f)\right] + \frac{2m}{\hbar^2}(E - V(x))\, f = 0$

donc : $\left[\begin{array}{l} \underset{\approx 0}{f''} - \theta'^2 f + \frac{p^2}{\hbar^2} f = 0 \\ \\ 2f'\theta' + \theta'' f = 0 \end{array}\right.$ $\Rightarrow$ $\left[\begin{array}{l} \theta' = \frac{p(x)}{\hbar} \text{ (1) car } \ominus \text{ à enclume} \\ \\ f\theta'' + 2f'\theta' = 0 \text{ (2)} \end{array}\right.$

avec particule $x \nearrow$ on $x \searrow$ contenu dans $p(x)$

b) (2) $\Rightarrow 2f'\theta' = -f\theta''$

$\Rightarrow \frac{f'}{f} = -\frac{1}{2}\frac{\theta''}{\theta'} \overset{(1)}{\Rightarrow} \frac{f'}{f} = -\frac{1}{2}\frac{p'}{p}$

soit $\ln\left(\frac{f}{cte}\right) = + \ln p^{-1/2}$

$\Rightarrow f(x) = cte \times p^{-1/2}$

donc $\boxed{\varphi(x) = cte\left[2m(E - V(x))\right]^{-1/4} e^{\frac{j}{\hbar}\int_0^x p(u)\,du}}$

② (H) $E < V(x)$ on recherche $\varphi(x) = f(x)\, e^{-\theta(x)}$. NB : evanescente

$f\theta'' + 2f'\theta' = 0$

$\left[\begin{array}{l} \varphi(x) = f(x)\, e^{-\theta(x)} \\ \varphi'(x) = f' e^{-\theta} - \theta' f e^{-\theta} \\ \varphi''(x) = f'' e^{-\theta} - \theta' f' e^{-\theta} + (-\theta'' f - \theta' f' + \theta'^2 f)\, e^{-\theta} \end{array}\right.$

ES $\Rightarrow -\frac{\hbar^2}{2m}\varphi''(x) + V(x)\varphi(x) = E\varphi(x) \Rightarrow \varphi''(x) - \underbrace{\frac{2m(V - E)}{\hbar^2}}_{K^2}\varphi(x) = 0$

donc : $\varphi''(x) - K^2\varphi(x) = 0$

$\Rightarrow f'' - \theta' f' - \theta'' f - \theta' f' + \theta'^2 f - K^2 f = 0$

$\Rightarrow -\underbrace{(f\theta'' + 2f'\theta')}_{\approx 0} + (\underset{\approx 0}{f''} + f\theta'^2) - K^2 f = 0$

$\Rightarrow f\theta'^2 = K^2 f \Rightarrow \theta' = \sqrt{K^2} = \sqrt{\frac{2m(V - E)}{\hbar^2}}$

$\Rightarrow \theta = \int_0^x \sqrt{\cdots}\, dx$

En reprenant finalement l'équation sur f :

$$f\theta'' + 2f'\theta' = 0$$

$$\Rightarrow \frac{f'}{f} = -\frac{1}{2}\frac{\theta''}{\theta'} \Rightarrow \ln\left(\frac{f}{cte}\right) = \ln \theta'^{-1/2}$$

$$\Rightarrow f = cte\ \theta'^{-1/2}$$

donc $$\varphi(x) = cte\left(\frac{2m(V-E)}{\hbar^2}\right)^{-1/4} e^{-\int_0^x \sqrt{\frac{2m(V(x)-E)}{\hbar^2}}\,dx}$$

donc : $$T_{WKB} = \left|\frac{\varphi(x_2)}{\varphi(x_1)}\right|^2 = \frac{e^{-2\int_0^{x_2}\sqrt{\cdots}\,dx}}{e^{-2\int_0^{x_1}\sqrt{}\,dx}}$$

$$\Rightarrow \boxed{T_{WKB} = e^{-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx}}$$